

# The not-so-effective mass of photons in a planar optical cavity

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It is a well established and understood fact that photons propagating in free space interact with the gravitational field, leading to well-known effects such as gravitational redshift or gravitational lensing. While these phenomena might give an impression that photons in free space have a sort of mass, this impression falls short upon considering their dispersion relation. In this letter we show that unlike in free space, when photons are brought to a stop within a planar cavity, they acquire a mass that cannot not be distinguished from that of a solid-state body freely moving in a bidimensional space, both from the inertial and gravitational point of view.

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The idea that a light beam should be “attracted” by massive astrophysical bodies like the sun was put forward for the first time by Johan Soldner in 1804 [1], using arguments just developed in Newton’s description of gravity. With the advent of general relativity at the dawn of the twentieth century, a quantitative description of light travelling in a gravity field has become available. The physics of light travelling within a gravitational field is nowadays well established and shows up in spectacular observable effects such as gravitational lensing [2, 3] or gravitational redshift. An experimental proof of the latter has been obtained for the first time in a celebrated experiment by Pound and Rebka in 1960 [4]. In this famous article, the authors emphasize the fact that such phenomena might give an impression that photons in free space have a sort of mass.

And yet, this impression falls short upon considering their dispersion relation, which characterizes the inertial mass of photons. Indeed in free space, a nonzero inertial photon mass would have several observable effects that have been thoroughly looked after experimentally. These effects are e.g. a deviation from the inverse square Coulomb potential for distances comparable with the photon Compton wavelength  $\lambda_c = \hbar/m_\gamma c$  [5], or a frequency dependant speed of light at ultra low frequencies [6]. The earlier effect has provided a still up-to-date, reliable upper limit for the photon mass of  $m_\gamma c^2 < 10^{-14} \text{eV}$  [7] as discussed in a recent review on the subject [8].

In this letter we show that this mass analogy does not fall short in the case of a wavepacket of light confined within the spacer of a conventional planar optical cavity, and that the photon “effective” mass, which is usually presented as a mathematical device to describe this situation, actually satisfies every criteria of a “true” mass, i.e. that associated with a solid body.

Inside an optical cavity, photons that propagate normally at the speed of light in free space, can be brought to a full stop, in a well defined region of space delimited by the cavity spacer. In the context of special relativity, since photons have a fixed non-zero energy, and since their translational degree of freedom is blocked by the

cavity, they formally acquire a rest mass  $m_\parallel$ . It is not an intrinsic mass of the bosonic field  $m_\gamma$  as discussed before, but it is instead acquired by putting photons of finite energy at rest. This interpretation is more profound than it is sometime believed, and the aim of this letter is to demonstrate that in addition to be a perfectly well defined mass from the inertial point of view it is also a gravitational mass, as expected from the equivalence principle [9].

Let us take as a starting point of this discussion the textbook dispersion of light in a planar (with flat mirrors of infinite surface) optical cavity. Owing to the translational invariance, the in-plane photon momentum  $\mathbf{k}_\parallel = (k_x, k_y)$  is a “good quantum number”. Applying the constructive interference condition of the field along the  $z$ -axis, upon reflection on the mirrors, one gets the usual cavity dispersion relation :

$$\hbar\omega = \sqrt{\left[\frac{\hbar\pi jc}{L}\right]^2 + (\hbar ck_\parallel)^2}, \quad (1)$$

where  $L$  is the cavity thickness,  $j$  a positive integer,  $\hbar\omega$  the photon energy, and  $\mathbf{p}_\parallel = \hbar\mathbf{k}_\parallel$  its in-plane momentum. This dispersion can be very simply rewritten as

$$[\hbar\omega]^2 = [m_\parallel c^2]^2 + [\hbar ck_\parallel]^2, \quad (2)$$

where

$$m_\parallel = \hbar\pi j/cL = E_0/c^2. \quad (3)$$

is the effective mass of the photon, and  $E_0$  is the photon rest energy. What is most frequently found in textbook in optics is that this dispersion resembles that of a free particle of mass  $m_\parallel$  as long as the kinetic term  $\hbar ck_\parallel \ll E_0$ . This is the so-called effective mass approximation. And yet, in the form of eq.(2) it is quite clear, and it has been noted before [10, 11], that if we account for special relativity, this dispersion matches exactly that of a relativistic particle of mass  $m_\parallel$ . The word “approximation” in “effective mass approximation” should thus be removed in the context of planar optical cavities. But

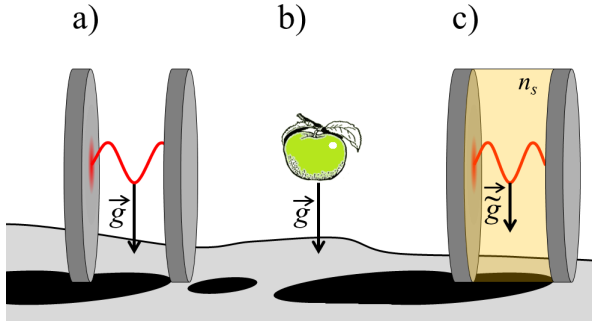


FIG. 1. Illustration of the formal identity between the free fall motion of a standing wavepacket of light in a vacuum filled (a) or dielectric filled (c) planar optical cavity and a classical non-relativistic object represented by an apple (b).

this analogy goes even further. Let us now consider the wave aspect of this problem of effective mass: the general dynamics of the scalar electric field in the cavity is governed by Helmholtz equation

$$\nabla^2 \Psi + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Psi = 0. \quad (4)$$

where  $\Psi(x, y, z)$  is the electric field. Assuming that the field has a well defined frequency  $\omega/2\pi$ , inside the planar cavity, the  $z$  dependence of the field is easily derived again from the constructive interference condition mentioned above  $\Psi(x, y, z) = u(x, y) \cos(j\pi z/L)$ . Eq.(4) thus simplifies into

$$\nabla_{\parallel}^2 u + \frac{1}{c^2} \left( \omega^2 - \frac{m_{\parallel}^2 c^4}{\hbar^2} \right) u = 0. \quad (5)$$

Strikingly, this expression matches exactly the Klein-Gordon equation, which is used to describe the dynamics of a quantum, spinless, relativistic massive particle in a bidimensional space (the cavity spacer). The photon mass given by this identity is again exactly  $m_{\parallel}$  without approximation. These properties underline quite clearly the fact that  $m_{\parallel}$  is not just an “effective” mass, it plays the exact role of a mass for the cavity photon from the inertial point of view. Note that in the non-relativistic limit, this equation simplifies into a Schrodinger equation, again with the mass  $m_{\parallel}$  in the kinetic term. By adding the leading non-linear term in this equation, we obtain the celebrated Gross-Pitaevskii equation that describes weakly interacting degenerate quantum gases [12] involving finite mass particles. Interestingly, quantum gases of dressed photons in planar cavities are also treated with a non-equilibrium version of this equation [13], with a great success in predicting their spatio-temporal dynamics as observed in experiments [14].

We now wish to examine whether this mass also plays a role from the gravitational point of view. In the framework of Newtonian gravitation, the answer is obviously

no since photons are bosonic particles of vanishing mass  $m_{\gamma} = 0$ . But obviously, the interaction between light and gravity has to be examined in the framework of general relativity. Let us assume a planar cavity oriented in the earth gravity field such that the cavity spacer plane is vertical (along  $y$ ), while  $x$  and the cavity axis  $z$  are in the horizontal plane. As we will discuss later on, we assume a very long cavity lifetime, like in super-conducting microwave cavities [15], or whispering gallery optical cavity [16], such that the light has enough time to interact with the gravity field.

the geometry of space-time inside the cavity is weakly perturbed by the gravity field and is well described by the Schwarzschild metric [17]. As discussed above, the electromagnetic field inside the cavity has no degree of freedom along  $z$ , its dynamics, including the  $z$  dependence of the phase, is entirely frozen by the confinement between the mirrors such that the group velocity along  $z$  is 0. The EM field can only acquire kinetic energy in the plane of the cavity under the action of gravity. Assuming the initial condition for the EM field  $k_{\parallel}(t=0) = 0$ , we can safely assume that the in-plane group velocity acquired during the photon lifetime always remain in the non-relativistic limit  $v_g(t) \ll c$ . In this limit the spatial part of the metric can be considered flat, while the proper time  $\tau$  takes the simple form

$$d\tau^2 = \left[ 1 - \frac{2GM}{rc^2} \right] dt^2 \quad (6)$$

in the weak gravitational field limit.  $r$  the radial distance from its center,  $t$  is the proper time when  $r \rightarrow \infty$ ,  $G$  is the gravitational constant, and  $M$  is the mass of the body generating the gravity field. This time dilation is well known in the context of massive object orbiting the earth, and has been tested in many different configuration [17]. In this context, the Helmholtz equation (4) involving this proper time

$$\nabla^2 \Psi + \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} \Psi = 0 \quad (7)$$

describes the dynamics of the standing wavepacket in the frame co-moving with it. For a fixed observer (in the laboratory frame, inside or outside the cavity), this time dilation appears as an inhomogeneous effective index  $n(r) \simeq 1 + \frac{GM}{rc^2}$ , or more specifically for an optical cavity at the surface of the earth,

$$n(y) = 1 + \frac{g}{c^2} (y_0 - y) \quad (8)$$

where  $g = 9.81 m.s^{-2}$  is the earth gravitational acceleration constant,  $y_0$  is the earth radius, and  $y$  the vertical co-ordinate measured from the ground. We will conveniently use  $n(y) = n_0 - gy/c^2$ . This procedure is better known in the context of the calculation of the deflection of light by astrophysical massive object (i.e. gravitational lensing).

It yields the same index of refraction within a factor of two. This is because unlike in a cavity, to describe light propagating in free space, the ultra-relativistic limit of the Schwarzschild metric has to be adopted. Indeed, in this limit the spatial part of the metric cannot be approximated to be Euclidean. The same factor of two and the same argument applies for the so-called gravitational redshift of light escaping a gravity well [17].

Let us now consider a photonic wavepacket inside the cavity centered in  $y = 0$  at time  $t = 0$ . When the gravitational index is taken into account, the rest energy  $m_{\parallel}n(y)c^2$  turns  $y$ -dependent. As discussed above, the kinetic energy assumes its non-relativistic form  $\hbar^2 k_y^2 / [2n(y)m_{\parallel}]$ . Since we assumed  $k_y(t = 0) = 0$ , the energy conservation yields

$$m_{\parallel}gy = \frac{\hbar^2 k_y^2}{2m_{\parallel}}, \quad (9)$$

where the gravitational correction of the kinetic term can be safely neglected. We recognize the textbook Newtonian gravitational energy of massive bodies on the left. The in-plane group velocity of the standing wavepacket is defined (in the non-relativistic limit) as  $v_g = (1/\hbar)\partial E_K / \partial k_y$  (where  $E_K$  is the kinetic energy) and using  $v_g = \partial y / \partial t$ , we can come up with a simple differential equation for the wave packet motion

$$\left(\frac{\partial y}{\partial t}\right)^2 = 2gy. \quad (10)$$

We finally obtain the equation of motion of the wavepacket

$$y(t) = -\frac{1}{2}gt^2. \quad (11)$$

One recognizes the typical parabolic dependence of a non-relativistic, Newtonian free fall. *A standing wavepacket of light within a planar cavity dipped in a gravitational field thus falls exactly at the same rate as a solid object of finite mass in vacuum.* This derivation unambiguously shows that the so-called effective mass  $m_{\parallel}$ , used in the context of optical cavities, is in fact a mass that correctly describes both the inertial and gravitational properties, in the Newtonian sense, of a wavepacket of light. Note that  $m_{\parallel}$  does not enter in eq.(11), like for usual massive object. It means that the cavity photon “effective mass” checks the equivalence principle as expected.

For practical purposes, we wish to examine now how the above considerations applies to the very common situation where the cavity spacer is filled with a solid-state medium of refractive index  $n_s > 1$ . Let us first examine the inertial properties of the wavepacket of light in such a cavity: The index of refraction modifies the dispersion relation into

$$[\hbar\omega]^2 = [\hbar\tilde{c}k_{\parallel}]^2 + [m_{s,\parallel}\tilde{c}^2]^2. \quad (12)$$

This is still a relativistic dispersion but with a renormalized speed of light  $\tilde{c} = c/n_s$  and a heavier rest mass  $m_{s,\parallel} = n_s^2 m_{\parallel}$ , assuming the same rest energy as in vacuum. With this simple transformation, we see that the rest mass still has a well defined inertial meaning, although in a space featuring a slower speed of light.

About the gravitational aspect of this mass, we can write again the Helmholtz equation, but this time including a fixed background index  $n_s$ . Within the same approximations, i.e. weak gravitational field and non-relativistic limit, the effective optical index now takes the form  $n(r) \simeq n_s(1 + \frac{GM}{rc^2})$ . For such an optical cavity situated at the surface of the earth, like for eq.(8), this index can be rewritten as

$$n(y) = n_s \left[ 1 + \frac{g}{c^2}(y_0 - y) \right]. \quad (13)$$

The total energy conservation involving the rest and the kinetic energy yields this time :

$$m_{s,\parallel} \frac{g}{n_s^2} y = \frac{\hbar^2 k_y^2}{2m_{s,\parallel}}, \quad (14)$$

like in eq.(9) this expression results in a Newtonian free fall

$$\left(\frac{\partial y}{\partial t}\right)^2 = 2\tilde{g}y. \quad (15)$$

but with a renormalized gravitational field  $\tilde{g} = g/n_s^2$ . The equivalence principle is thus well checked also for a wavepacket free falling inside a solid state optical cavity. However unlike in a vacuum cavity, the gravitational acceleration of the wavepacket is reduced by the medium, that acts as a kind of drag force for the wavepacket of light, as pointed out already in the context of light propagation in transparent moving media [18]. Interestingly, owing to translational invariance of the planar cavity, the in-plane momentum of the light  $k_{\parallel}$  is conserved when it leaks out of the optical cavity. This has two interesting consequences: (i) since energy is also conserved, the wavepacket dynamics as observed from outside the microcavity exactly maps that inside the cavity; (ii) this dynamics thus can be accessed experimentally by e.g. carrying out angle-resolved measurement of the leaking light [19] or by interferometric measurements.

Observing this free fall experimentally is quite challenging; and yet, unlike classical solid bodies, we have the advantage of having access to the phase  $\phi(y, t)$  of the wavepacket. Its vertical gradient  $\partial\phi/\partial y = v_g m_{s,\parallel} / \hbar$  is proportional to the group velocity and independent from the medium optical index; it indeed simplifies into

$$\frac{\partial\phi}{\partial y} = \frac{\omega_0 g t}{c^2}. \quad (16)$$

A possible such interferometric measurement of Newtonian free fall of light is thus described in Fig.2.a: since

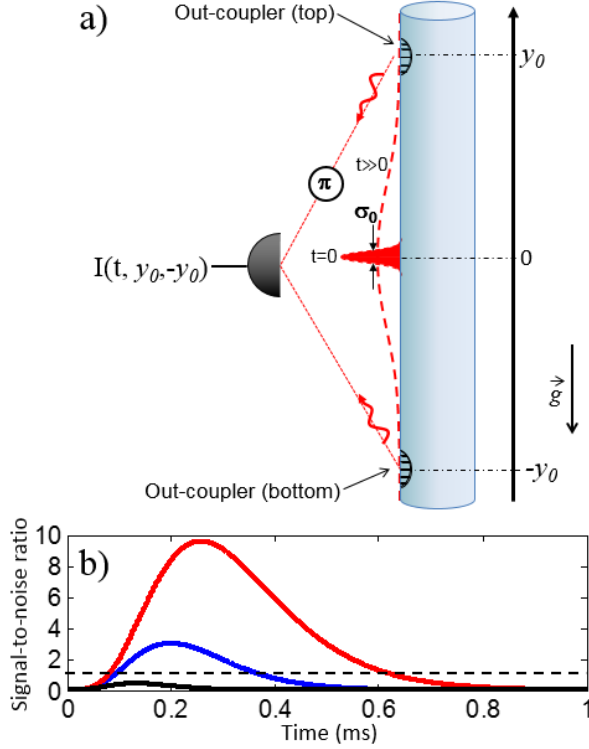


FIG. 2. a) Experimental setup for the measurement of Newtonian free fall in a cylindrical optical cavity. The WGMC has a diameter in the millimeter range in order to reach giant quality factor [20]. The following experimental parameters have been chosen to simulate  $S_n(t)$ , the signal-to-noise shown in b): light wavelength  $\lambda = 1064\text{nm}$ , initial wavepacket size  $\sigma_0 = 10\text{cm}$ , out-coupling height  $y_0 = 50\text{cm}$ , average laser power of  $1\text{mW}$ , an overall detection efficiency of  $10^{-3}$  is assumed, refractive index  $n_s = 1.43$  ( $\text{CaF}_2$ ), integration time  $T = 3600\text{s}$ , and quality factors  $Q = 3 \times 10^{10}$  (black),  $5 \times 10^{10}$  (blue) and  $7 \times 10^{10}$  (red).  $S_n = 1$  is marked as a horizontal dashed line.

the results presented in this letter are equally valid and easily transposed for cavity modes with only one degree of freedom, we propose to use dielectric cylinders in which the light is free to propagate along the main axis (with a scalar momentum  $k_a$ ), and confined into whispering gallery modes in the transverse direction. As illustrated in Fig.2.a, this whispering gallery modes cylinder (WGMC) is oriented vertically such that a wavepacket of light will undergo gravitational freefall along the main axis. For the light to build up a noticeable gravity induced phase difference, a giant quality factor is required. In this context, ultra pure materials like  $\text{CaF}_2$  or  $\text{MgF}_2$  are of particular interest, as they have been used to achieve whispering gallery mode structures displaying quality factor in excess of  $10^{10}$  [20, 21], i.e. close to that required in this experiment as is shown by the simulation below.

The experiment would be carried out in the following way: at  $t = 0$  a spatially Gaussian pulse of light (of

initial spatial width  $\sigma_0$  along  $y$ , is injected with zero axial momentum, in one of the whispering gallery modes by evanescent coupling. We then take advantage of the slow (tunable), long range, free expansion undergone by the Gaussian wavepacket in both directions of the WGMC. Some grating-like diffracting elements are placed at both ends of the waveguide (in  $y_0$  and  $-y_0$ ) in order to locally couple the light out of the WGMC. These two beams are then sent to interfere on a photodiode. A phase retardation is inserted into one of the paths such that at  $t \simeq 0$  the phase difference between both paths is exactly  $\pi$ . While this free expansion is usually symmetric, the phase gradient of eq.(16) builds up on top of it due to the gravity field. As a result, while the interference is fully destructive at early times, a non-zero signal appears at long time, due to the gravity induced phase rotation, i.e. to the free fall. Taking all these effects into account, the time dependent interference signal measured by the photodiode thus reads

$$I(t, y_0, -y_0) = \exp\left(-\frac{\omega_0 t}{Q} - \frac{y_0^2}{\sigma^2(t)}\right) \times (1 - \cos[\Delta\phi(y_0, t)]), \quad (17)$$

where the first term in the exponential accounts for the losses, and  $\sigma^2(t) = \sigma_0^2 + c^2 t^2 / (2\omega_0 n_s^2 \sigma_0)$  is the vertical free expansion of the mode of initial spatial width  $\sigma_0$ . The last term accounts for the gravity induced phase rotation  $\Delta\phi(y_0, t) = \omega_0 g t^2 y_0 / c^2$  between points  $y_0$  and  $-y_0$ . Assuming a photon shot-noise limited measurement, the simulated signal-to-noise ratio  $S_n = \sqrt{I(t, y_0, -y_0) T} / \hbar \omega_0$  is plotted in Fig.2.b for realistic experimental parameters (in particular a quality factor  $Q > 3 \times 10^{10}$ , cf. caption for details). We find that after  $T = 1$  hour integration time and an average laser power of  $1\text{mW}$ , a measurable free fall signal can be achieved. Note that a continuous wave version of this experiment would give a similar result. The major challenge in this experiment is the fabrication of a WGMC with negligible fluctuations of the diameter, that could otherwise cause the free fall to slow stop or even bounce back.

In this letter, we have thus shown that in “effective mass”, a notion routinely used to describe the dispersion of the light in planar (or cylindrical) cavities, “effective” should be dropped. Indeed as photons are brought to a full stop in a cavity, they indeed acquire a mass in the usual sense of the word, both from the inertial and the gravitational point-of-view. In particular we predict that in such a cavity, a photon undergoes non-relativistic Newtonian free fall, an effect which is different in magnitude and behaviour from gravitational redshift or deflection of the light in free space, both effects being ultra-relativistic in essence. We finally proposed a realistic experiment where this Newtonian free fall of light in a cavity might be measurable.

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